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Finding volumes between a quadric surface and a plane

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Abstract

In this paper, we give a summary of methods of finding the volume of a solid bounded between a quadric surface and a plane, which are extensions to [2,4]. *We include analytical methods, and applications of Divergence and Stokes' Theorems.*

1 Introduction

In this paper, we give descriptions of finding the volume bounded by a quadric surface and a plane. Unless otherwise is specified, we only discuss the cases where there is a finite volume for the bounded solid. In particular, we will focus on the surfaces of an ellipsoid, a hyperboloid of two sheets and a paraboloid. We usually start with a general introduction of a method and lead to a particular example. In this paper, we give several analytical methods using various coordinate systems where most students learn in a multivariable Calculus class. We often see the needs of change of variables and change of bases to transform an arbitrary quadric surface into a standard form [1] before applying integration techniques, the methods described here are accessible to those who have learned Linear Algebra and multivariable Calculus. We assume readers are familiar with the Divergence and Stokes' Theorems and we will also describe how we may apply these two principles to find the volumes of bounded solid.

We use the following five coordinate systems:

– initial coordinate system X, in which the quadric and the plane are given;

– canonical coordinate system X', used principal axis in which the quadric is in standard form [1, Table 3.5-2, 3.5-7];

– "natural" coordinate system X'', in which the origin is located in the center of a cross section of the quadric and the plane and the axes are oriented along the axes of the section;

– "harmonized" system of spherical coordinate (ρ, ϑ, ϕ) connected with natural coordinate system X''. The *x*-axis for "harmonized" system is of $\vartheta=0$, the *y*-axis is of $\phi=0, \vartheta=\frac{\pi}{2}$, and *z*-axis is of $\phi=\frac{\pi}{2}, \vartheta=\frac{\pi}{2}$;

– two-dimensional coordinates (θ, φ), used for parametric representations of quadrics [1, 3.5-22, 3.5-24,3.5-25].

Each of the volumes is calculated by the following six methods.

Method 1. Integration by quadratures using the coordinates X''.

Method 2. Numerical integration of the distance from point on quadric to the plane using the coordinates X''.

Method 3. Numerical integration of the distance from point on quadric to the origin of "natural" coordinate system X'' with using "harmonized" system of spherical coordinate (ρ, ϑ, ϕ).

Method 4. Numerical integration using the Divergence Theorem in the coordinates (θ, φ).

Method 5. Numerical integration with the use of Stokes' Theorem in the coordinates (θ, φ).

Method 6. Integration with the use of discontinuous integrand in the coordinates (θ, φ).

2 General Description of a Quadric surface and a Plane

Since our main focus is on finding the volume bounded by a quadric surface and a plane. We first consider the equation of the quadric surface in a matrix form:

$$\vec{X}^T A \vec{X} + 2\vec{B}\cdot\vec{X} + a_{00} = 0, \tag{1}$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}, \vec{B} = (a_{10}, a_{20}, a_{30}), \vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, a_{ij} are real numbers, $i, j = 0,1,2,3$, and $a_{11} > 0$. Alternatively, we sometimes write a quadric surface in the scalar form of:

$$f(x, y, z) = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2a_{10}x + 2a_{20}y + 2a_{30}z + a_{00} = 0. \tag{2}$$

Also, we consider the equation of the plane in the forms of

$$\vec{N}\cdot\vec{X} + N_0 = 0, \quad \vec{n}\cdot\vec{X} + n_0 = 0, \text{ and } g(x, y, z) = n_1 x + n_2 y + n_3 z + n_0 = 0, \tag{3}$$

where $\vec{N} = (N_1, N_2, N_3)$ is a normal vector to the plane, N_0 is a constant, $\vec{n} = (n_1, n_2, n_3)$, and $n_i = \dfrac{N_i}{\sqrt{N_1^2 + N_2^2 + N_3^2}}$ for $i = 0,1,2,$ and 3.

2.1. Transformation of the quadrics equations. We are interested in those solids bounded by quadric surfaces and a plane produce finite volumes. The quadric surface we discuss in this paper will be an ellipsoid, an elliptic paraboloid, a hyperboloid of two sheets or a cone; however, we will leave it to reader to explore the case when quadric surface is a cone. We classify the quadric surfaces by using the following scenarios:

Case 1. If $I = a_{11} + a_{22} + a_{33}$, $J = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{33} & a_{13} \\ a_{13} & a_{11} \end{vmatrix} > 0$, $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = 0$, and $\vec{B}\cdot\vec{V}_3 \neq 0$, where $\vec{V}_3$ is the normalize eigenvector corresponds to $\lambda_3 = 0$. Then the quadric is an elliptic paraboloid. Indeed, we have

$$\lambda_1=\frac{I+\sqrt{I^2-4J}}{2}, \quad \lambda_2=\frac{I-\sqrt{I^2-4J}}{2}, \quad \lambda_3=0, \tag{4}$$

and the corresponding eigenvectors to λ_1 and λ_2, in case of three different eigenvalues, are

$$\vec{v}_1=\begin{pmatrix} a_{12}a_{23}-(a_{22}-\lambda_1)a_{13} \\ a_{12}a_{13}-(a_{11}-\lambda_1)a_{23} \\ (a_{11}-\lambda_1)(a_{22}-\lambda_1)-a_{12}^2 \end{pmatrix} \text{ and } \vec{v}_2=\begin{pmatrix} a_{12}a_{23}-(a_{22}-\lambda_2)a_{13} \\ a_{12}a_{13}-(a_{11}-\lambda_2)a_{23} \\ (a_{11}-\lambda_2)(a_{22}-\lambda_2)-a_{12}^2 \end{pmatrix}, \tag{5}$$

respectively. If we have a repeated eigenvalue, we use (5) for the eigenvector $\vec{v}_1$ associated with the single eigenvalue, and the vector $\vec{v}_2$ can be choose to be any vector perpendicular to $\vec{v}_1$. If one of these vectors is $\vec{0}$, we rename corresponding λ as λ_3. We remind the readers to make necessary renumbering of vectors in various special cases. We normalize $\vec{v}_i$ and write $\vec{V}_i=\frac{\vec{v}_i}{|\vec{v}_i|}$ for $i=1,2$. We set $\vec{V}_3=[\vec{V}_1\times\vec{V}_2]$ and form the matrix $R=(\vec{V}_1,\vec{V}_2,\vec{V}_3)$. If we set $x_0=\frac{\vec{B}\cdot\vec{V}_1}{\lambda_1}$, $y_0=\frac{\vec{B}\cdot\vec{V}_2}{\lambda_2}$, $z_0=\frac{a_{00}-\lambda_1 x_0^2-\lambda_2 y_0^2}{2\vec{B}\cdot\vec{V}_3}$, and $\vec{X}_0=\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$,

then the substitution $\vec{X}=R(\vec{X}'-\vec{X}_0)$ will transform the original quadric into

$$-\frac{\lambda_1}{2\vec{B}\cdot\vec{V}_3}(x')^2-\frac{\lambda_2}{2\vec{B}\cdot\vec{V}_3}(y')^2=z'. \tag{6}$$

We note that the signs of λ_1 and λ_2 are the same since $J=\lambda_1\lambda_2>0$, which gives us an alternative way of categorizing the quadric to be an **elliptic paraboloid**. We denote $a=\sqrt{\left|\frac{2\vec{B}\cdot\vec{V}_3}{\lambda_1}\right|}$, and $b=\sqrt{\left|\frac{2\vec{B}\cdot\vec{V}_3}{\lambda_2}\right|}$. If the sign of $\vec{B}\cdot\vec{V}_3$ coincides with the sign of λ_1, we change the direction of the axis z' to the opposite direction. We obtain the equation of the paraboloid in the form of $\frac{(x')^2}{a^2}+\frac{(y')^2}{b^2}=z'$. In matrix form we obtain the following equation:

$$\vec{X}'^T A'\vec{X}'+2\vec{B}'\cdot\vec{X}'=0, \text{ where } A'=\begin{pmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \vec{B}'=(0,0,-\frac{1}{2}). \tag{7}$$

Case 2. Suppose $\Delta\neq 0$ and the signs of λ_1, λ_2 and λ_3 are the same and they are different from the sign of $a_{00}-\vec{B}A^{-1}\vec{B}^T$. Then such a quadric corresponds to an **ellipsoid**. In particular, the substitution $\vec{X}=R\vec{X}'-A^{-1}\vec{B}^T$ will transform the original ellipsoid into

$$\lambda_1(x')^2+\lambda_2(y')^2+\lambda_3(z')^2-\vec{B}A^{-1}\vec{B}^T+a_{00}=0. \tag{8}$$

We denote $a=\sqrt{\dfrac{\vec{B}A^{-1}\vec{B}^T-a_{00}}{\lambda_1}}, b=\sqrt{\dfrac{\vec{B}A^{-1}\vec{B}^T-a_{00}}{\lambda_2}}$, and $c=\sqrt{\dfrac{\vec{B}A^{-1}\vec{B}^T-a_{00}}{\lambda_3}}$. As a result, we obtain the equation in the form of $\dfrac{(x')^2}{a^2}+\dfrac{(y')^2}{b^2}+\dfrac{(z')^2}{c^2}=1$. In matrix form we obtain the following equation:

$$\vec{X}'^T A'\vec{X}'+2\vec{B}'\cdot\vec{X}'+a'_{00}=0,\ \text{where}\ A'=\begin{pmatrix} a^{-2} & 0 & 0\\ 0 & b^{-2} & 0\\ 0 & 0 & c^{-2}\end{pmatrix},\quad \vec{B}'=\vec{0},\ \text{and}\ a'_{00}=-1. \tag{9}$$

Case 3. Suppose $\Delta\neq 0$ and not all of the signs of λ_1, λ_2 and λ_3 are the same, among the numbers λ_1, λ_2, λ_3 and $a_{00}-\vec{B}A^{-1}\vec{B}^T$, one number differs in sign from the other three numbers. Such quadric corresponds to a **hyperboloid of two sheets.** We assume the sign of λ_1 differs from the signs of λ_2, λ_3, and $a_{00}-\vec{B}A^{-1}\vec{B}^T$. We name this situation as sub-case 1. The substitution $\xi_1=\dfrac{x'}{a}, \xi_2=\dfrac{y'}{b}, \xi_3=\dfrac{z'}{c}$, $a=\sqrt{\dfrac{\vec{B}A^{-1}\vec{X}_0-a_{00}}{\lambda_1}}$, $b=\sqrt{\dfrac{a_{00}-\vec{B}A^{-1}\vec{X}_0}{\lambda_2}}$, and $c=\sqrt{\dfrac{a_{00}-\vec{B}A^{-1}\vec{X}_0}{\lambda_3}}$ will transform equation (8) into the form of $\xi_1^2-\xi_2^2-\xi_3^2=1$; that is, to the canonical equation of the hyperboloid of two sheets. In matrix form we reach the following equation:

$$\vec{X}'^T A'\vec{X}'+2\vec{B}'\cdot\vec{X}'+a'_{00}=0,\ \text{where}\ A'=\begin{pmatrix} a^{-2} & 0 & 0\\ 0 & -b^{-2} & 0\\ 0 & 0 & -c^{-2}\end{pmatrix},\quad \vec{B}'=\vec{0},\ \text{and}\ a'_{00}=-1. \tag{10}$$

Exercise. Obtain the same results in other sub-cases within case 3.

The other cases beyond cases 1,2 and 3 will not produce finite volumes for the solids, which we will not discuss here.

2.2. Equation for the curve of intersection – natural coordinates. Let us show that the intersection curve of the quadric and the plane is an ellipse. The equation of the quadric surface, we use in the form of (1), is equivalent to (7), (9) and (10). The equation of the plane will be written in the form of $\vec{m}\cdot\vec{X}'+m_0=0$, and $|\vec{m}|=1$.

2.2.0. We adopt the coordinate system in which the equation of the cross section has the forms of $x''=0$ and $a_{23}y''z''=0$. Let the origin of the coordinate system x'', y'', z'', $\vec{X}''=\begin{pmatrix}x''\\ y''\\ z''\end{pmatrix}$ be at point $\vec{A}_0$ which will be defined later, the basis in this coordinate system contains vectors $\vec{m}, \vec{U}$, and $\vec{m}\times\vec{U}$. The corresponding transformation is $\vec{X}'=r\vec{X}''+\vec{A}_0$, where $r=(\vec{m}, \vec{U}, \vec{m}\times\vec{U})$. Vector $\vec{U}$ lies on the plane of the cross section, and it will be defined explicitly later.

The substitution in the equation (1) yields

$$(\vec{X}'')^T rA'r^T\vec{X}''+2(\vec{A}_0^T A'+\vec{B})r\vec{X}''+\vec{A}_0^T A'\vec{A}_0+2\vec{B}\cdot\vec{A}_0+a_{00}=0, \tag{11}$$

and the substitution in the plane yields $\vec{m}\cdot(r\vec{X}''+\vec{A}_0)+m_0=0$. The element of the matrix $r^T A' r$ with the (2,3) position, second row and third column of $r^T A' r$, is 0 or $(r^T A' r)_{23}=\vec{U}^T A'[\vec{m}\times\vec{U}]=0$, since this determines the coefficient of the product $y''z''$.

We find the basis vector $\vec{U}$ by solving the following simultaneous equations:

$$\begin{cases} \vec{m}\cdot\vec{U}=0, \\ \vec{U}^T A'[\vec{m}\times\vec{U}]=0, \\ |\vec{U}|=1. \end{cases} \tag{12}$$

Each of the roots of the quadratic equation (12) corresponds to one of the axes of the quadric surface, we may use either solution for further investigation. These solutions corresponds to $\vec{U}$ and $\vec{V}=\vec{m}\times\vec{U}$. We get the components of $\vec{U}^T=(U_x, U_y, U_z)$ by using the following equations: $(a^2-b^2)c^2 m_3^2 m_2 U_y^2+(b^2c^2(m_2^2-m_3^2)-a^2b^2(m_1^2+m_2^2)+a^2c^2(m_1^2+m_3^2))U_y+b^2(c^2-a^2)m_2=0$, $U_x=-\dfrac{1+m_2U_y}{m_1}$, and $U_z=\dfrac{1}{m_3}$. We next normalize the vector $\vec{U}$.

With information from the vector $\vec{U}$, it allows us to define the coefficient of $(y'')^2$ to be $l_2=(r^T A' r)_{22}=\vec{U}^T A'\vec{U}$ and the coefficient of $(z'')^2$ to be $l_3=(r^T A' r)_{33}=\vec{V}^T A'\vec{V}$. We define the origin of coordinates $\vec{A}_0$ to be on the plane of the cross section $\vec{m}\cdot\vec{A}_0+m_0=0$. We make the coefficients of y'' and of z'' equal to zero. That is $((\vec{A}_0^T A'+\vec{B}')r)_2=0$, and $((\vec{A}_0^T A'+\vec{B}')r)_3=0$. Then $\vec{A}_0$ defines the center for the ellipse of the cross section. The constant term of the equation (11) in the coordinates x'', y'', z'' is equal to $a_0''=\vec{A}_0^T A'\vec{A}_0+2\vec{B}'\vec{A}_0+a'_{00}$. The semi–axes of the ellipse are respectively

$$a_e=\sqrt{\frac{-a_0''}{l_2}} \quad \text{and} \quad b_e=\sqrt{\frac{-a_0''}{l_3}}. \tag{13}$$

Hence, the cross section is an ellipse if and only if the signs of the numbers l_2 and l_3 are the same and differ from the sign of a_0''.

Let us write the equations defining the intersection curve in each of the cases. For convenience, we use the point $\vec{T}$ to denote the point where the tangent plane to the quadric is parallel to the given plane; obtaining such $\vec{T}$ is possible due to [2]. We have one such point for the paraboloid. We have two such points for the ellipsoid and the hyperboloid of two sheets, and the points are symmetrical about the origin. Simple calculations yield the following results.

2.2.1. We use the substitution $\vec{X}=R(\vec{X}'-\vec{X}_0)$ for the paraboloid in the equation (2) and yields

$$\vec{m}\cdot\vec{X}'+m_0=0, \quad \text{where} \quad \vec{m}=\vec{n}R, m_i=\vec{n}\cdot\vec{V}_i, m_0=n_0-R\vec{n}\cdot\vec{X}_0. \tag{14}$$

The center of the ellipse $\vec{A}_0$ and the point $\vec{T}$ have the following coordinates respectively:

$$x'_A=x'_T=-\frac{a^2m_1}{2m_3}, \quad y'_A=y'_T=-\frac{b^2m_2}{2m_3}, \quad z'_T=\frac{a^2m_1^2+b^2m_2^2}{4m_3^2}, \quad \text{and} \quad z'_A=-\frac{m_0}{m_3}+2z'_T. \tag{15}$$

The semi-axes of the ellipse are a_e and b_e respectively: $a_e=\sqrt{\dfrac{-a_0''}{l_2}}$ and $b_e=\sqrt{\dfrac{-a_0''}{l_3}}$,

where $$-a_0''=z'_T-\frac{m_0}{m_3},\quad l_2=\frac{U_x^2}{a^2}+\frac{U_y^2}{b^2},\quad l_3=\frac{V_x^2}{a^2}+\frac{V_y^2}{b^2},\ \text{and}\ l_2 l_3=\frac{m_3^2}{a^2 b^2}. \tag{16}$$

The area of the ellipse is $$S_e(m_0)=\pi\, a_e b_e=\frac{-a_0''\pi}{\sqrt{l_2 l_3}}=\pi\, a\, b\,\frac{z'_T-\dfrac{m_0}{m_3}}{m_3}. \tag{17}$$

The coordinates of any point of the quadric in the solid of intersection have the following standard vector form:

$$\vec{X}'=\vec{T}+a_e(t)\vec{U}\cos\varphi+b_e(t)\vec{V}\sin\varphi+(0,0,t),\, a_e(t)=\sqrt{\frac{t}{l_2}},\, b_e(t)=\sqrt{\frac{t}{l_3}},\ \text{where}\ t\in[0,z_A-z_T]. \tag{18}$$

2.2.2. For the ellipsoid, the substitution $\vec{X}=R\vec{X}'-A^{-1}\vec{B}^T$ in the equation (2) gives

$$\vec{m}\vec{X}'+m_0=0,\ \text{where}\ \vec{m}=\vec{n}R,\ \text{and}\ m_0=\vec{n}\vec{X}_0+n_0. \tag{19}$$

The center of the ellipse $\vec{A}_0$ and the point $\vec{T}$ have the following coordinates:

$$x'_A=-\frac{m_0 m_1 a^2}{w^2},\quad y'_A=-\frac{m_0 m_2 b^2}{w^2},\ \text{and}\ z'_A=-\frac{m_0 m_3 c^2}{w^2},\ \text{where}\ w^2=m_1^2a^2+m_2^2b^2+m_3^2c^2,$$

$$x'_T=-\frac{m_1 a^2}{w},\quad y'_T=-\frac{m_2 b^2}{w},\quad z'_T=-\frac{m_3 c^2}{w},\ \text{and}\ \vec{T}'=-\vec{T}. \tag{20}$$

The semi-axes of the ellipse are a_e and b_e respectively: $a_e=\sqrt{\dfrac{-a_0''}{l_2}}$ and $b_e=\sqrt{\dfrac{-a_0''}{l_3}}$,

where $-a_0''=1-\dfrac{m_0^2}{w^2}$,

$$l_2 l_3=\frac{w^2}{a^2b^2c^2(m_1^2+m_2^2+m_3^2)},\quad l_2=\frac{U_x^2}{a^2}+\frac{U_y^2}{b^2}+\frac{U_z^2}{c^2},\ \text{and}\ l_3=\frac{V_x^2}{a^2}+\frac{V_y^2}{b^2}+\frac{V_z^2}{c^2}. \tag{21}$$

The area of the ellipse is

$$S_e(\mu_0)=\pi\, a_e b_e=\frac{-a_0''\pi}{\sqrt{l_2 l_3}}=\pi\, a\, b\, c\,\frac{\sqrt{m_1^2+m_2^2+m_3^2}}{w}(1-\mu_0^2),\ \text{where}\ \mu_0=\frac{m_0}{w}. \tag{22}$$

The coordinates of any point of the quadric in the solid of intersection have the following standard vector form:

$$\vec{X}'=-\mu\vec{T}+a_e(t)\vec{U}\cos\varphi+b_e(t)\vec{V}\sin\varphi,\, a_e(t)=\sqrt{\frac{1-\mu^2}{l_2}},\, b_e(t)=\sqrt{\frac{1-\mu^2}{l_3}},\ \text{and}\ \mu\in[-1,\mu_0]. \tag{23}$$

2.2.3. For the hyperboloid of two sheets, if the sign of λ_1 differs from the signs of λ_2,λ_3 and $a_{00}-\vec{B}A^{-1}\vec{B}^T$, then the substitution $\vec{X}=R\vec{X}'-A^{-1}\vec{B}^T$ in the equation (2) gives equation (19).

The center of the ellipse, $\vec{A}_0$ and the point $\vec{T}$ have the following coordinates:

$$x'_A=-\frac{m_0 m_1 a^2}{w^2},\quad y'_A=\frac{m_0 m_2 b^2}{w^2},\ \text{and}\ z'_A=\frac{m_0 m_3 c^2}{w^2},\ \text{where}\ w^2=m_1^2a^2-m_2^2b^2-m_3^2c^2,$$

$$x'_T=\frac{m_1 a^2}{w},\qquad y'_T=-\frac{m_2 b^2}{w},\qquad z'_T=-\frac{m_3 c^2}{w},\quad \text{and}\quad \vec{T}'=-\vec{T}. \tag{24}$$

The semi-axes of the ellipse are a_e and b_e respectively: $a_e=\sqrt{\dfrac{a_0''}{-l_2}}$ and $b_e=\sqrt{\dfrac{a_0''}{-l_3}}$, where

$$a_0''=\frac{m_0^2}{w^2}-1,\qquad -l_2=\frac{-U_x^2}{a^2}+\frac{U_y^2}{b^2}+\frac{U_z^2}{c^2},\qquad -l_3=\frac{-V_x^2}{a^2}+\frac{V_y^2}{b^2}+\frac{V_z^2}{c^2},\quad \text{and}\quad l_2 l_3=\frac{w^2}{a^2b^2c^2(m_1^2+m_2^2+m_3^2)}. \tag{25}$$

The area of the ellipse is $S_e(\mu_0)=\pi\, a_e b_e=\dfrac{a_0''\pi}{\sqrt{l_2 l_3}}=\pi\, a b c\dfrac{\sqrt{m_1^2+m_2^2+m_3^2}}{w}(\mu_0^2-1),\ \mu_0=\dfrac{m_0}{w}.$ (26)

The coordinates of any point of the quadric in the solid of intersection have standard vector form:

$$\vec{X}'=-\mu\vec{T}+a_e(t)\vec{U}\cos\varphi+b_e(t)\vec{V}\sin\varphi,\ a_e(t)=\sqrt{\frac{\mu^2-1}{-l_2}},\ b_e(t)=\sqrt{\frac{\mu^2-1}{-l_3}},\quad \text{and}\quad \mu\in[\mu_0,-1]. \tag{27}$$

The other cases will not produce finite cross section, which we will not discuss here.

Remarks. We make the following observations in view of the equations (22-27):

1. A set the of the cross sections of the given quadric by parallel planes is a family of the similar ellipses.

2. We may say that:

a) a diameter of the paraboloid is a beam, parallel to its axis and beginning at point $\vec{T}$;

b) a diameter of the ellipsoid is a segment passing through its center and connecting points $\vec{T}'$ and $\vec{T}$;

c) a diameter of the hyperboloid of two sheets is a beam, passing through its center and outgoing into the infinity from points $\vec{T}'$ and $\vec{T}$ and lying on the straight line $\vec{T}\vec{T}'$.

The diameter contains the centers of the ellipses obtained from the intersection of the quadric and the family of planes parallel to the given plane, which touches the quadric at the point T. The Figure 1 illustrates the statements 1 and 2. It shows the solids cut off by the plane from the paraboloid (Fig. 1a), the ellipsoid (Fig. 1b) and the hyperboloid of two sheets (Fig.1c). By varying the parameter m_0, we construct family of parallel planes and observe the center of the elliptical cross-section slides in the direction along the diameter.

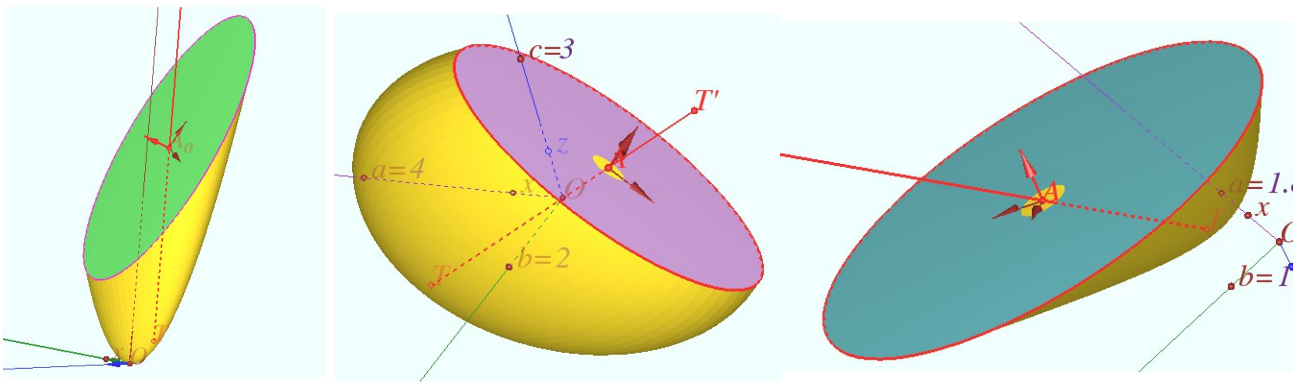

Figure 1. Solids cut off by the plane from the paraboloid (Fig. 1a), the ellipsoid (Fig. 1b) and the hyperboloid of two sheets (Fig.1c). (Plot produced with [GInMA])

2.3. Equation for the curve of intersection-classic coordinates

In the description of the quadric surfaces, we use the canonical variables X' as initial variables and rewrite the equation in the internal variables for the surface.

2.3.1. For the paraboloid described by the equations (7) and (14), we make the substitution

$$\begin{cases} x'=a\,t\cos\varphi, \\ y'=b\,t\sin\varphi, \\ z'=t^2, \\ t\geq 0. \end{cases} \tag{28}$$

We transform the plane equation into $$t^2-2t_s t+\frac{m_0}{m_3}=0,\, t_s=\frac{x'_A}{a}\cos\varphi+\frac{y'_A}{b}\sin\varphi. \tag{29}$$

The solution of this equation is $t=t_s\pm\sqrt{t_s^2-\frac{m_0}{m_3}}$. If the vertex of the paraboloid X_0 belongs to the intersection solid, then we have a unique positive solution $t=t_s+\sqrt{t_s^2-\frac{m_0}{m_3}}$ for any value of φ. If X_0 does not belong to the solid of intersection, then there are two positive solutions $t>0$, which exist only if φ satisfying the condition of $t_s^2\geq\frac{m_0}{m_3}$.

2.3.2. For the ellipsoid described by the equations (8) and (19) we make the following substitution

$$\begin{cases} x'=a\sin\theta\cos\varphi, \\ y'=b\sin\theta\sin\varphi, \\ z'=c\cos\theta, \end{cases} \tag{30}$$

and the intersection curve between the ellipsoid and the plane is

$$(a\,m_1\cos\varphi+b\,m_2\sin\varphi)\sin\theta+c\,m_3\cos\theta+m_0=0. \tag{31}$$

Under the condition $|m_3 c|>|m_0|$, the above equation has a unique positive solution for θ when $\varphi\in[0,2\pi]$,

$$\sin\theta=\frac{m_0^2-c^2m_3^2-t_s m_0-c\,m_3\sqrt{c^2m_3^2+t_s^2-m_0^2}}{c^2m_3^2+t_s^2-t_s m_0+m_3 c\sqrt{c^2m_3^2+t_s^2-m_0^2}},\qquad \cos\theta=\frac{(c\,m_3+\sqrt{c^2m_3^2+t_s^2-m_0^2})(t_s-m_0)}{c^2m_3^2+t_s^2-t_s m_0+m_3 c\sqrt{c^2m_3^2+t_s^2-m_0^2}},$$

and $$\theta(\varphi)=\frac{\pi}{2}-\arccos\frac{t_s}{\sqrt{c^2m_3^2+t_s^2}}-\arccos\frac{-m_0}{\sqrt{c^2m_3^2+t_s^2}},\quad \text{where}\quad t_s=a\,m_1\cos\varphi+b\,m_2\sin\varphi. \tag{32}$$

2.3.3. For the hyperboloid of two sheets described by the equations (10) and (19), we make the substitution

$$\begin{cases} x=a\cosh t, \\ y=b\cos\varphi\sinh t, \\ z=c\sin\varphi\sinh t. \end{cases} \tag{33}$$

The curve of intersection between the hyperboloid and the plane is

$$(b\,m_2\cos\varphi+c\,m_3\sin\varphi)\sinh t+a\,m_1\cosh t+m_0=0. \tag{34}$$

Under the condition $-\dfrac{m_0}{m_1}>a$, the equation (34) has a unique positive solution for *t* for any

$$\varphi\in[0,2\pi]:\quad t=\ln\frac{1+\tau}{1-\tau},\quad \text{where}\quad \tau=\frac{t_s-\sqrt{t_s^2+m_0^2-a^2m_1^2}}{m_0-m_1a},\quad \text{and}\quad t_s=b\,m_2\cos\varphi+c\,m_3\sin\varphi. \quad (35)$$

3 Examples

We consider the following three randomly selected solids. Let us first consider a paraboloid whose vertex belongs to the solid of intersection.

Example 1 *We consider the paraboloid given by the equation of* $\vec{X}^T A\vec{X}+2\vec{B}\vec{X}+a_{00}=0$, *where* $A=\begin{pmatrix}5&4&7\\4&6&1\\7&1&243/14\end{pmatrix}$, $\vec{B}=\begin{pmatrix}-2\\-3\\7\end{pmatrix}$, $a_{00}=-7$, *and* $\vec{X}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ *or we may write the paraboloid in the form of* $f(x,y,z)=5x^2+8xy+6y^2+14xz+2yz+\dfrac{243}{14}z^2-4x-6y+14z-7=0$, *and consider the plane in the form of* $g(x,y,z)=x-3y+2z-2=0$. *Let us name the solid bounded by the quadric surface and the plane* S_1. *We note that the eigenvalues for the paraboloid are* $\dfrac{397+\sqrt{36145}}{28}$, $\dfrac{397-\sqrt{36145}}{28}$, *and* 0, *respectively. The eigenvectors for the paraboloid are* $\vec{V}_1\approx\begin{pmatrix}0.4317\\0.1745\\0.8850\end{pmatrix}$, $\vec{V}_2\approx\begin{pmatrix}0.3846\\0.8519\\-0.3555\end{pmatrix}$ *and* $\vec{V}_3\approx\begin{pmatrix}-0.8159\\0.4938\\0.3006\end{pmatrix}$, *respectively. If we write* $f(x,y,z)$ *in the canonical form by using the canonical variable* X', *we get* $\left(\dfrac{x'}{a}\right)^2+\left(\dfrac{y'}{b}\right)^2=z'$, $a\approx0.46373$, *and* $b\approx0.78120$. *The equation of the plane when using the canonical variable* X' *is* $m_1x'+m_2y'+m_3z'+m_0=0$ *where* $m_1\approx0.4485$, $m_2\approx-0.77026$, $m_3\approx-0.4533$, *and* $m_0\approx-2.51798$. *The center of the elliptical cross section is at the point* $\vec{A}_0=(\dfrac{21217}{6241},-\dfrac{4919}{6241},-\dfrac{11746}{6241})$. *The calculations above are obtained from Maple* (*see* [7],[8]).

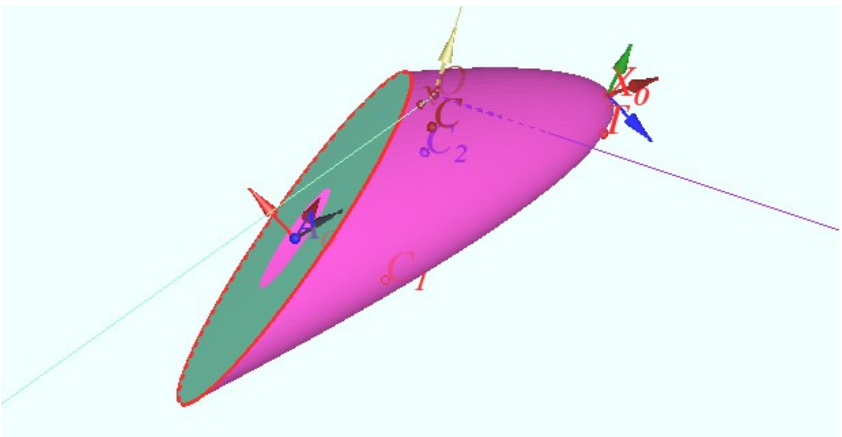

Figure 2. Solid S_1 formed by the paraboloid and the plane.

Next we consider a solid formed by an ellipsoid and a plane as follows:

Example 2 *Consider the ellipsoid of the form* $\vec{X}^T A\vec{X} + 2\vec{B}\vec{X} + a_{00} = 0$, *where*

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 5 & 2 \\ 0 & 2 & 2 \end{pmatrix}, \vec{B} = \begin{pmatrix} -2 \\ 3 \\ -3 \end{pmatrix}, a_{00} = 5, \quad \text{and} \quad \vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$ *or we may write the surface as*

$f(x, y, z) = x^2 - 2xy + 5y^2 + 4yz + 2z^2 - 4x + 6y - 6z + 5 = 0$, *and consider the semi-space* $g(x, y, z) = 3x - 4y + 3z - 21 \le 0$. *We note that the eigenvectors for the ellipsoid are* $\vec{V}_1 \approx \begin{pmatrix} -0.1724 \\ 0.8877 \\ 0.4271 \end{pmatrix}$, $\vec{V}_2 \approx \begin{pmatrix} -0.6318 \\ 0.2332 \\ -0.7392 \end{pmatrix}$, *and* $\vec{V}_3 \approx \begin{pmatrix} -0.7558 \\ -0.3971 \\ 0.5207 \end{pmatrix}$, *respectively. The center of the ellipsoid is at the point* $\vec{X}_0 = (0, \ -2, \ 3.5)$, *the semi–axes for the ellipsoid are* $a \approx 1.3667$, $b \approx 2.8982$, *and* $c \approx 4.9226$, *respectively. The equation for the ellipsoid is* $\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 + \left(\frac{z'}{c}\right)^2 - 1 = 0$, *and for the plane is* $m_1 x' + m_2 y' + m_3 z' + m_0 = 0$, *where* $m_1 \approx 0.4777$, $m_2 \approx 0.8654$, $m_3 \approx -0.1514$, *and* $m_0 \approx 0.4287$. *Let us name the solid of this intersection* S_2. *The center of the cross section of the solid, an ellipse, is at the point* $\vec{A}_0 = \left(\frac{5}{43}, -\frac{96}{43}, \frac{168}{43}\right)$. *The calculations above are obtained from Maple* (*see* [7],[8]).

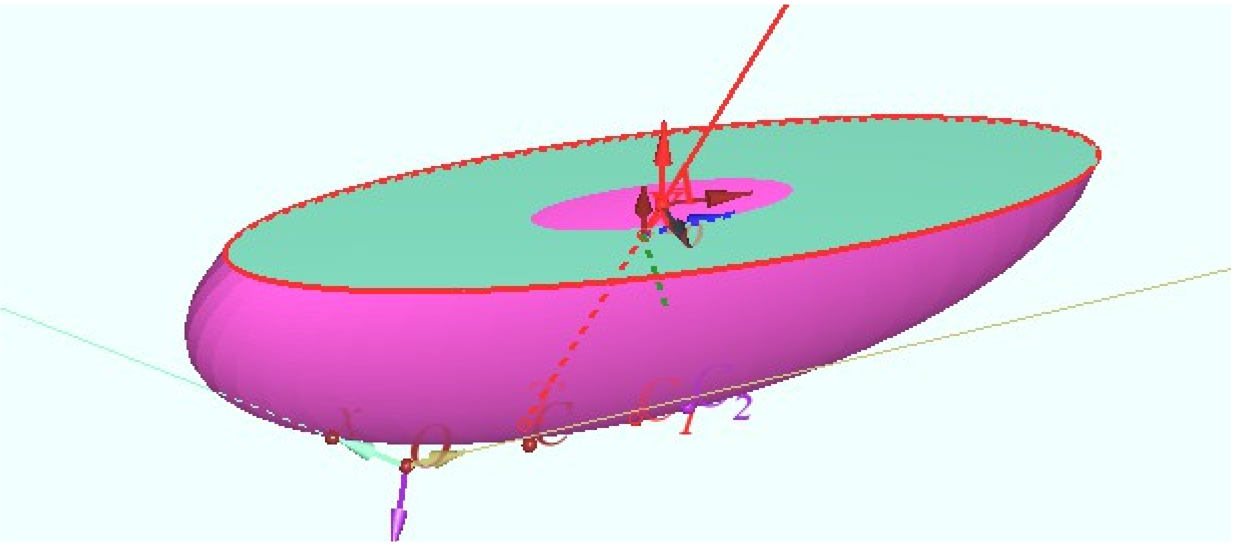

Figure 3. Solid S_2 formed by the ellipsoid and the plane.

Let us consider the following solid formed by a hyperboloid of two sheets and a plane as follows:

Example 3 *We consider the hyperboloid of two sheets of the form* $\vec{X}^T A\vec{X} + 2\vec{B}\cdot\vec{X} + a_{00} = 0$,

where $$A = \begin{pmatrix} 5 & -1 & -4 \\ -1 & -3 & 2 \\ -4 & 2 & -6 \end{pmatrix}, \vec{B} = \begin{pmatrix} -2 \\ -1 \\ -7 \end{pmatrix}, a_{00} = -14, \quad \text{and} \quad \vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$ *or we may write the surface as*

$f(x,y,z)=5x^2-2xy-3y^2-8xz-6z^2+4yz-4x-2y-14z-14=0$ *and consider the plane of* $g(x,y,z)=9x-7y-z+19=0$. *The eigenvectors are* $\vec{V}_1 \approx \begin{pmatrix} -0.9319 \\ 0.1650 \\ 0.3230 \end{pmatrix}$, $\vec{V}_2 \approx \begin{pmatrix} -0.2544 \\ -0.9320 \\ -0.2580 \end{pmatrix}$, *and* $\vec{V}_3 \approx \begin{pmatrix} 0.2585 \\ -0.3226 \\ 0.9106 \end{pmatrix}$, *respectively. The center of the quadric is at the point* $\vec{X}_0=(-\frac{3}{5},-\frac{29}{35},-\frac{73}{70})$. *If the variable* X' *is used, the equations for the hyperboloid of two sheets becomes* $\left(\frac{x'}{a}\right)^2-\left(\frac{y'}{b}\right)^2-\left(\frac{z'}{c}\right)^2-1=0$, *where the semi axes for the hyperboloid are* $a \approx 0.84365, b \approx 1.31066$, *and* $c \approx 0.77172$, *respectively. The plane can be written as* $m_1 x'+m_2 y'+m_3 z'+m_0=0$ *where* $m_1 \approx -0.8619, m_2 \approx 0.3925, m_3 \approx 0.3210$, *and* $m_0 \approx 1.7861$. *Let us name the solid of such intersection* S_3. *The center of the elliptical cross section is at the point* $\vec{A}_0=(-\frac{5343}{796},-\frac{4789}{796},\frac{140}{199})$. *The calculations above are obtained from Maple* (*see* [7],[8]).

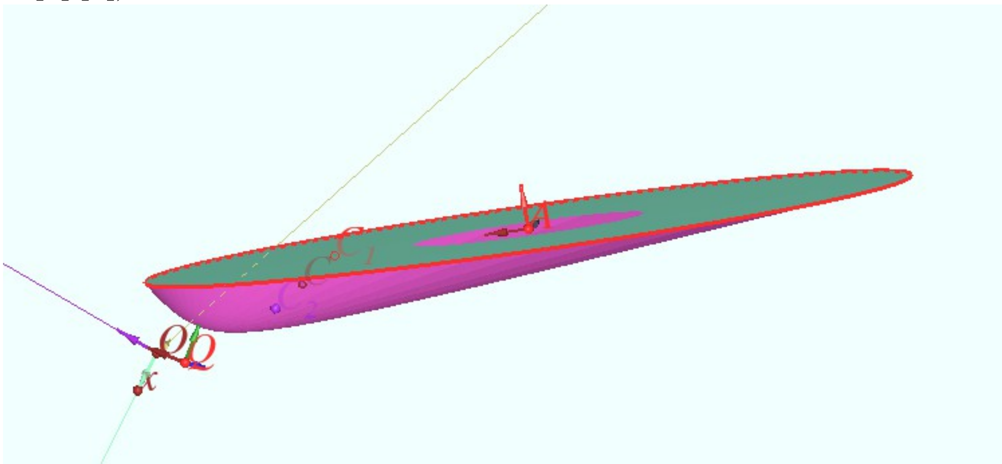

Figure 4. Solid S_3 formed by the hyperboloid and the plane.

4 Calculation methods

4.1 Volume and Natural Coordinates

Let the solid be bounded by the given quadric and the plane of $m_1 x'+m_2 y'+m_3 z'+m_0=0$. We consider the set of cross sections of the solid by parallel planes of $m \in (m_T, m_0]$, where $m_T=-m_1 x'_T-m_2 y'_T-m_3 z'_T$. The set contains similar ellipses of the area $S_e(m)$, whose centers $\vec{A}_m$ are located on the segment $\vec{A}_m \in (\vec{T}, \vec{A}_0]$. The point $\vec{T}$ is excluded. In this point the ellipse degenerates into a point. The volume of the investigated solid is easy to find by integrating the area with respect to the coordinate z'. It is noted that for a paraboloid we see $m_3 \neq 0$, and yet $m_3=0$ for other two solids. We perform integration using different coordinate when $m_3=0$, which does not affect the result. The volume of the solid bounded by the quadric and the plane is:

$$V=\int_{z'_T}^{z'_A} S_e(m)\,dz'=\frac{\partial z'}{\partial m}\int_{m_T}^{m_0} S_e(m)\,dm=-\frac{1}{m_3}\int_{m_T}^{m_0} S_e(m)\,dm. \tag{36}$$

4.1.1 Volume of the solid bounded by the paraboloid and the plane. We consider a paraboloid of the form $\frac{(x')^2}{a^2}+\frac{(y')^2}{b^2}=z'$ and a plane of the form $m_1x'+m_2y'+m_3z'+m_0=0$. It follows from equation (17) that the area of the cross section is $S_e(m)=\pi ab\frac{z'_T-\frac{m}{m_3}}{m_3}$. Furthermore, we see from equation (36) that:

$$V=-\frac{1}{m_3}\int_{m_T}^{m_0} S_e(m)\,dm=\frac{\pi ab}{m_3^2}\int_{m_T}^{m_0}(m_T-m)\,d\,m=\frac{\pi ab}{2}\left(\frac{m_T}{m_3}-\frac{m_0}{m_3}\right)^2,$$

$$V=\frac{\pi ab}{2}\left(\frac{m_0}{m_3}+\frac{a^2m_1^2+b^2m_2^2}{4m_3^2}\right)^2. \tag{37}$$

An Exercise. Find the volume of S_1 by using the method mentioned in this section.

Answer: The volume of the solid S_1, calculated by using formula (37), is approximately 20.80907893464 (*see Maple worksheet in* [9]).

4.1.2. Let the solid be bounded by an ellipsoid of $\frac{(x')^2}{a^2}+\frac{(y')^2}{b^2}+\frac{(z')^2}{c^2}=1$ and a plane of an equation $m_1x'+m_2y'+m_3z'+m_0=0$. Then $S_e(m)=\pi abc\sqrt{\frac{m_1^2+m_2^2+m_3^2}{w}}\left(1-\frac{m^2}{w}\right)$. The volume of the solid bounded by the ellipsoid and given plane is:

$$V=-\frac{1}{m_3}\int_{m_T}^{m_0} S_e(m)\,dm=\pi abc\sqrt{m_1^2+m_2^2+m_3^2}\int_{m_{0T}}^{m_0}\left(1-\frac{m^2}{w}\right)d\left(\frac{m}{\sqrt{w}}\right),$$

$$V=\frac{\pi abc}{3}(2+3\mu-\mu^3),\ \mu=\frac{m_0}{\sqrt{w}}\in[-1;1]. \tag{38}$$

The volume of the solid S_2, calculated by formula (38), is approximately 50.496501830226. We note the values of $V(\vec{T})=0$ and $V(\vec{T}')=\frac{4}{3}\pi abc$. $\vec{T}(\vec{T}')$ are poles in which polar planes are parallel to given plane ([1, 3,5-8,c]) (*see Maple worksheet in* [9]).

4.1.3. Let the solid be bounded by a hyperboloid of $\frac{(x')^2}{a^2}-\frac{(y')^2}{b^2}-\frac{(z')^2}{c^2}=1$ and a plane $m_1x'+m_2y'+m_3z'+m_0=0$. Then $V=\frac{\pi abc}{3}(2+3\mu-\mu^3),\ \mu=-\frac{|m_0|}{\sqrt{w}}<0.$ (39)

The volume of the solid S_3 calculated using formula (39) is approximately 46.9249144931, $\mu\approx-3.96665310075$ (*see Maple worksheet in* [9]).

4.1.4 Mapping Ellipsoid to Sphere

To calculate the volume of the two pieces of an ellipsoid intersected by a plane, we may transform an ellipsoid to a sphere by stretching. Specifically, we turn the ellipsoid into the sphere by stretching along perpendicular axes, which coincide with the principal axes of the ellipsoid. The secant plane by stretching along perpendicular axes goes into the plane. Let us consider the distance from the secant plane to the center of the sphere l, and the ratio between l and the radius of the sphere ρ, say $k=\frac{l}{\rho}$. Then the ratio of the volume for the segments of the sphere is equal to

$$K=\frac{2-3k+k^3}{2+3k-k^3}. \tag{40}$$

It is clear that the ratio K under compression or stretching will stay the same. If each of the principal axes of the ellipsoid is stretched with the coefficient $\sqrt{\lambda_i}$, or $\tilde{x}'=x'\sqrt{\lambda_1}$, $\tilde{y}'=y'\sqrt{\lambda_2}$, and $\tilde{z}'=z'\sqrt{\lambda_3}$, then the ellipsoid becomes a sphere with the radius $\rho=\sqrt{\vec{B}A^{-1}\vec{B}^T-a_{00}}$ and the volume $V_0=\frac{4\pi}{3}(\vec{B}A^{-1}\vec{B}^T-a_{00})^{3/2}$. Subsequently, the volume of the ellipsoid is

$$V=\frac{4\pi}{3\sqrt{\Delta}}(\vec{B}A^{-1}\vec{B}^T-a_{00})^{3/2}, \quad \text{where } \Delta=\lambda_1\lambda_2\lambda_3. \tag{41}$$

To find k, we use the equation of the plane of the form (19) $m_1x'+m_2y'+m_3z'+m_0=0$, where $m_i=\vec{n}\vec{V}_i$ and $m_0=n_0-\vec{n}A^{-1}\vec{B}^T$. As a result of stretching we get

$$\frac{m_1}{\sqrt{\lambda_1}}\tilde{x}'+\frac{m_2}{\sqrt{\lambda_2}}\tilde{y}'+\frac{m_3}{\sqrt{\lambda_1}}\tilde{z}'+m_0=0. \tag{42}$$

Therefore, the distance from the plane to the center of the sphere; that is the distance from the plane (42) to the origin is

$$l=\frac{|m_0|}{\sqrt{\frac{m_1^2}{\lambda_1}+\frac{m_2^2}{\lambda_2}+\frac{m_3^2}{\lambda_3}}}. \tag{43}$$

An Exercise. Find the volume of the solid S_2 by using the method mentioned in this section.

Answer: We calculate the volume of the ellipsoid S_2 by using formula (41), and it is approximately 81.67805252196. The volumes of two ellipsoidal pieces calculated by formulas (40) and (43) are approximately 50.496501830226 and 31.181550691739. Since the solid S_2 contains the center of the ellipsoid, the volume of S_2 should be more than half of the ellipsoid, and hence the volume of the solid S_2 is approximately 50.496501830226 (*see Maple worksheet in* [9]).

4.2 Volume and Spherical Coordinates

Our goal now is to find the volume of the solid of the intersection between the quadric surface and the plane by using spatial spherical coordinates.

Let point $\vec{A}_0$ be located on the given plane. We use coordinate system X'' with the origin at $\vec{A}_0$ and axis x'' perpendicular to the plane. For considered solids S_1, S_2, and S_3, it is convenient to choose a coordinate system with origin at the center of the elliptical cross section and

the axes oriented along the ellipse axes. In general, the point $\vec{A}_0$ can be chosen arbitrary. We recall that the equation (1) of the quadric is
$\vec{X}''^T r A r^T \vec{X}'' + 2(\vec{A}_0^T A + \vec{B}) r \vec{X}'' + \vec{A}_0^T A \vec{A}_0 + 2\vec{B}\vec{A}_0 + a_{00} = 0,$ the equation of the plane (2) is $x'' = 0$. We use the following spherical coordinate:

$$\begin{cases} x'' = \rho\cos\vartheta, \\ y'' = \rho\sin\vartheta\sin\phi, \\ z'' = \rho\sin\vartheta\cos\phi. \end{cases} \tag{44}$$

We note the Jacobian $J = \rho^2 \sin\vartheta$, and by substituting into the quadric equation, we reach a quadratic equation with respect to $\rho(\vartheta, \phi)$. In other words, we have

$$k_2(\vartheta, \phi)\rho^2 + k_1(\vartheta, \phi)\rho + k_0 = 0, \tag{45}$$

where $k_0 = \vec{A}_0^T A \vec{A}_0 + 2\vec{B}\vec{A}_0 + a_{00}$. We note that the paraboloid and the ellipsoid have the value of $k_0 > 0$ and unique positive solution of the equation (45) exists as follows:

$$\rho(\vartheta, \phi) = \frac{2k_0}{\sqrt{k_1^2(\vartheta, \phi) - 4k_0 k_2(\vartheta, \phi)} - k_1(\vartheta, \phi)}. \tag{46}$$

In the case of the hyperboloid of two sheets, we observe that $k_0 < 0$ and we obtain two positive solutions of the quadric equation (45) which correspond to two different sheets of the hyperboloid. We need the solution that corresponds to the desired bounded volume.

If the basis vector of the axis x'' is directed toward the surface, the volume is

$$V = \int_0^{2\pi} d\phi \int_0^{0.5\pi} \sin\vartheta\, d\vartheta \int_0^{\rho(\vartheta,\phi)} \rho^2 d\rho = \frac{1}{3}\int_0^{2\pi} d\phi \int_0^{0.5\pi} \rho^3(\vartheta, \phi)\sin\vartheta\, d\vartheta. \tag{47}$$

Otherwise, in the integral over ϑ, the upper limit should be changed to $-\frac{\pi}{2}$.

The choice of point $\vec{A}_0$ is not unique. It is sufficient that point is located on the surface of the intersection solid. Its position is indicated by the sign of $k_0 = \vec{A}_0^T A \vec{A}_0 + 2\vec{B} \cdot \vec{A}_0 + a_{00}$. We may assume that the calculation accuracy increases if point is located near the center of the intersection curve of the quadric surface and the plane.

We make calculations using Maple **student[simpson]** with 256 points (*see Maple worksheet in* [8]). We use 15 digits in all calculations. We obtain that the volume of the solid S_1 is approximately 20.8090792. The volumes for the solid S_2 and S_3 are approximately 50.4965018300 and 46.924952, respectively.

The method has an interesting trick in the numerical solution. If point $\vec{A}_0$ is on the intersecting plane, but outside the solid (outside the intersecting ellipse) and we use standard Maple solving command **solve,** Maple return a complex solution for ρ and may obtain a complex number as the result for the volume calculation. This is surprising for an inexperienced user. But the correct result is equal to zero and also may be calculated using Maple. The authors apply the following trick to perform the calculations in cases where complex solutions occur: we add the inequalities that are identically satisfied (for example $x + 1111 > 0$) into the simultaneous equations which we solve using Maple. After this Maple calculations with **solve** command, the answers often become correct.

Users should be noted that there is a significant difference between a theoretical solution and computational one, when Maple command **solve** is used.

4.3 Volume and Divergence Theorem

We first discuss a general strategy of how we may use the Divergence theorem to find the volume of the solid bounded by a quadric surface and a plane. We assume the followings:

(1) the standard form equation of the quadric $f(x', y', z')=0$ uses the canonical coordinate system X' and principal axis,

(2) the equation of the plane is $\vec{m}\cdot\vec{X}'+m_0=0$, where $\vec{m}=(m_1, m_2, m_3)$ is a unit vector,

(3) the transformation of parametric representations to canonical coordinate system is either $(t, \varphi)\to X'$ [1, 3.5-25] or $(\theta, \varphi)\to X'$ [1, 3.5-22, 3.5-24].

(4) the equation of the intersection curve uses the coordinates $t(\varphi)$ or $\theta(\varphi)$.

In this subsection, we discuss how we may find the volumes of the regions bounded by a quadric and a plane by using the following equation from Divergence theorem:

$$\iiint_V \nabla\vec{T}\,dV=\iint_S \vec{T}\cdot\vec{ds}, \tag{48}$$

where $\vec{ds}=\begin{pmatrix} dy'dz' \\ dz'dx' \\ dx'dy' \end{pmatrix}$. We define a vector field $\vec{T}$ such that $\begin{cases}\nabla\vec{T}=C_0, \\ \vec{m}\cdot\vec{T}=0,\end{cases}$ where $C_0\neq 0$. It follows from $\nabla\vec{T}=C_0$ that $\iiint_V \nabla\vec{T}\,dV=C_0\iiint_V dV=C_0 V$, which is the volume of the bounded solid multiplied by C_0.

On the other hand, since $\vec{m}\cdot\vec{T}=0$, and the vector $\vec{ds}$ (on the plane) is collinear with $\vec{m}$, we see that $\iint_{plane} \vec{T}\cdot\vec{ds}=0$. Therefore the volume of the solid of intersection is the surface integral of $\vec{T}$ over the given quadric. In other words we have

$$V=\frac{1}{C_0}\iint_S \vec{T}\cdot\vec{ds}. \tag{49}$$

Next we attempt to find the simplest form for $\vec{T}\cdot\vec{ds}$.

Suppose we write $\vec{T}$ in the form of

$$\vec{T}=(c_{11}x'+c_{12}y'+c_{13}z', c_{21}x'+c_{22}y'+c_{23}z', c_{31}x'+c_{32}y'+c_{33}z'). \tag{50}$$

We get $\nabla\vec{T}=c_{11}+c_{22}+c_{33}=C_0$, $\vec{m}\cdot\vec{T}=\tilde{m}_1 x'+\tilde{m}_2 y'+\tilde{m}_3 z'=0$, where

$$\tilde{m}_1=m_1 c_{11}+m_2 c_{21}+m_3 c_{31}=0, \quad \tilde{m}_2=m_1 c_{12}+m_2 c_{22}+m_3 c_{32}=0, \text{ and}$$

$$\tilde{m}_3=m_1 c_{13}+m_2 c_{23}+m_3 c_{33}=0. \tag{51}$$

We have 3 equations for 9 variables and one inequality $C_0\neq 0$. We consider the following scenarios.

Case 1. If the surface is a paraboloid, by using the change of variables equation (28), we get $\vec{ds}$ $\vec{ds}=\begin{pmatrix} ds_1 \\ ds_2 \\ ds_3 \end{pmatrix}$ with the components $ds_1=\frac{\partial(y', z')}{\partial(t, \varphi)}dt\,d\varphi=\frac{t^2\cos\varphi\,dt\,d\varphi}{\vec{B}\cdot\vec{V}_3\sqrt{\lambda_2}}$,

$$ds_2=\frac{\partial(z', x')}{\partial(t, \varphi)}dt\,d\varphi=\frac{t^2\sin\varphi\,dt\,d\varphi}{\vec{B}\cdot\vec{V}_3\sqrt{\lambda_1}}, \quad\text{and}\quad ds_3=\frac{\partial(x', y')}{\partial(t, \varphi)}dt\,d\varphi=\frac{t\,dt\,d\varphi}{\sqrt{\lambda_1\lambda_2}}. \tag{52}$$

We proceed to find $\vec{T}\cdot\vec{ds}=\vec{T}_p\cdot\vec{ds}$, where we use $\vec{T}_p$ for the vector field $\vec{T}$ on the paraboloid. It follows from the equations (51) and (52) that $\vec{T}_p\cdot\vec{ds}$ is a polynomial of t. We observe that the right side of equation (49) has nine terms of c_{ij}. We can find seven terms of c_{ij} to be equal to zero. Furthermore, we find $C_0=1$ and $\vec{T}_p=(-\frac{m_3}{m_1}z',0,z')$, and note that

$\vec{T}_p\cdot\vec{ds}=\frac{m_3 t^4\cos\varphi\, dt\, d\varphi}{2m_1(\vec{B}\vec{V}_3)^2\sqrt{\lambda_2}}-\frac{t^3\, dt\, d\varphi}{2\vec{B}\vec{V}_3\sqrt{\lambda_1\lambda_2}}$. Consequently, we get

$$V=\iint_S \vec{T}_p\cdot\vec{ds}=\frac{m_3\int_0^{2\pi} t(\varphi)^5\cos\varphi\, d\varphi}{10m_1(\vec{B}\cdot\vec{V}_3)^2\sqrt{\lambda_2}}-\frac{\int_0^{2\pi} t(\varphi)^4\, d\varphi}{8\vec{B}\cdot\vec{V}_3\sqrt{\lambda_1\lambda_2}}, \tag{53}$$

where $t(\varphi)$ is the solution to the equation (29) for the curve of intersection.

Case 2. In the case of an ellipsoid, we use the change of variables equation (30), to get $\vec{ds}$ with the components $ds_1=\frac{\partial(y',z')}{\partial(\varphi,\theta)}=bc\cos\varphi\sin^2\theta$, $ds_2=\frac{\partial(z',x')}{\partial(\varphi,\theta)}=ac\sin\varphi\sin^2\theta$, and

$$ds_3=\frac{\partial(x',y')}{\partial(\theta,\varphi)}=ab\sin\theta\cos\theta. \tag{54}$$

Let $\vec{T}_e$ be the vector field $\vec{T}$ on the ellipsoid. We find $C_0=2m_1^2a^2-m_2^2b^2-m_3^2c^2$,

$\vec{T}_e=(-(m_2^2b^2+m_3^2c^2)x'-m_1a^2(m_2y'+m_3z'),m_1(m_2b^2x'+m_1a^2y'),m_1(m_3c^2x'+m_1a^2z'))$.

$\vec{T}_e\vec{ds}=(m_1^2a^2+m_2^2b^2+m_3^2c^2)abc\sin^3\theta\cos^2\varphi\, d\theta\, d\varphi-m_1^2a^3bc\sin\theta\, d\theta\, d\varphi$. We get

$$\iint_S \vec{T}_e\cdot\vec{ds}=(m_1^2a^2+m_2^2b^2+m_3^2c^2)abc\int_0^{2\pi}\cos^2\varphi\, d\varphi\int_{\theta(\varphi)}^{\pi}\sin^3\theta\, d\theta-m_1^2a^3bc\int_0^{2\pi}d\varphi\int_{\theta(\varphi)}^{\pi}\sin\theta\, d\theta=$$

$$=(m_1^2a^2+m_2^2b^2+m_3^2c^2)abc\int_0^{2\pi}\cos^2\varphi(\cos\theta(\varphi)+\frac{2-\cos^3\theta(\varphi)}{3})d\varphi-m_1^2a^3bc\int_0^{2\pi}(1+\cos\theta(\varphi))d\varphi, \tag{55}$$

where $\theta(\varphi)$ is the solution to the equation (32) for the curve of intersection.

Case 3. In the case of a hyperboloid of two sheets, we use the change of variables equation (33), to get $\vec{ds}$ with the components $ds_1=\frac{\partial(y',z')}{\partial(t,\varphi)}=bc\sinh t\cosh t$,

$$ds_2=\frac{\partial(z',x')}{\partial(t,\varphi)}=-ac\sinh^2 t\cos\varphi,\quad\text{and}\quad ds_3=\frac{\partial(x',y')}{\partial(t,\varphi)}=-ab\sinh^2 t\sin\varphi. \tag{56}$$

Let $\vec{T}_h$ be the vector field $\vec{T}$ on the hyperboloid. We obtain $C_0=2m_1^2a^2+m_2^2b^2+m_3^2c^2$,

$\vec{T}_h=((m_2^2b^2+m_3^2c^2)x'-m_1a^2(m_2y'+m_3z'),m_1(-m_2b^2x'+m_1a^2y'),m_1(-m_3c^2x'+m_1a^2z'))$.

$\vec{T}_h\vec{ds}=(-m_1^2a^2+m_2^2b^2+m_3^2c^2)abc\cosh^2 t\sinh t\, dt\, d\varphi+m_1^2a^3bc\sinh t\, dt\, d\varphi$. We get

$$\iint_S \vec{T}_h\cdot\vec{ds}=(-m_1^2a^2+m_2^2b^2+m_3^2c^2)abc\int_0^{2\pi}d\varphi\int_0^{t(\varphi)}\cosh^2 t\sinh t\, dt+m_1^2a^3bc\int_0^{2\pi}d\varphi\int_0^{t(\varphi)}\sinh t\, dt=$$

$$=(-m_1^2a^2+m_2^2b^2+m_3^2c^2)abc\int_0^{2\pi}\frac{\cosh^3 t(\varphi)-1}{3}d\varphi+m_1^2a^3bc\int_0^{2\pi}(\cosh t(\varphi)-1)d\varphi. \tag{57}$$

We now apply the equations (53), (55), and (57) to find the volume bounded by the quadric and the plane as follows:

Example 4 *We consider the scenario of the volume discussed in* **Example 1**. *We substitute the solution* (29) *for the intersection curve in the equation* (53). *We choose the unique positive solution, taking into account the signs of the numbers* n_0 *and* $\frac{n_0 n_3}{\vec{B}\cdot\vec{V}_3}$. *As a result of calculations we obtain the volume of the solid* S_1 *to be about* 20.80907893464 (*see Maple worksheet in* [9]).

Example 5 *We would like to find the volume bounded by the ellipsoid and the plane. We consider the scenario of the volume discussed in* **Example 2**. *We substitute the solution* (32) *for the intersection curve in the equation* (53). *As a result of the calculations, we find the volume of the solid* S_2 *is approximately* 50.496501830226. *The calculation runs faster if we use* $\vec{T}_e$ *and* $\theta(\varphi)$. *In this case Maple is able to perform the symbolic integration command* ***int.*** *We use different possible Maple* ***int*** *subcommands such as* ***method = _d01akc*** *and* ***method = _Ncrule***. *They lead to the same value of the volume but takes different time for calculations.* (*See Maple worksheet in* [9]).

Example 6 *We consider the scenario of the volume discussed in* **Example 3**. *We substitute the solution* (35) *for the intersection curve in the equation* (57). *As a result of calculations we find the volume of the solid* S_3 *to be about* 46.9249144928 (*see Maple worksheet in* [9]).

We observe from the calculations made in this section 4.3 that the vector field $\vec{T}$ can be any vector field satisfying $\begin{cases}\nabla\vec{T}=C_0, C_0\neq 0,\\ \vec{m}\cdot\vec{T}=0.\end{cases}$ In the first calculation we use the simplest form for the vector field

$$\vec{T}=\left(0, n_3 y' - n_3\frac{(1-n_3)}{n_2}z', -n_2 y' + (1-n_3)z'\right). \tag{58}$$

In this case, we do not need to load the Maple package, **student[simpson]** for calculation. We note that the computation time will be much less when using Maple command, **int**, after obtaining $\vec{T}_p, \vec{T}_e,$ and $\vec{T}_h$ individually.

4.4 Volume and the Stokes' Theorem–Part 1

Our focus now is to apply the Stokes' theorem to find the volume of the solid bounded by the quadric surface and the plane. Assume that we know the canonical equation of the quadric, the equation of the plane $\vec{m}\cdot\vec{X}'+m_0=0$, where $\vec{m}=(m_1, m_2, m_3)$ is a unit vector, and the equation of the curve of intersection. We consider the equation of

$$\iiint_V \nabla\times\vec{F}\, dV=\iint_S \vec{ds}\times\vec{F}, \quad \text{where} \quad \vec{ds}=\begin{pmatrix} dy'dz' \\ dz'dx' \\ dx'dy' \end{pmatrix}. \tag{59}$$

We define a vector field $\vec{F}$ satisfying $\begin{cases}\nabla\times\vec{F}=\vec{const},\\ \vec{m}\times\vec{F}=\vec{0}.\end{cases}$ For example, $\vec{F}=\begin{pmatrix} m_1 x' \\ m_2 x' \\ m_3 x' \end{pmatrix}$. Then

$$\nabla\times\vec{F}=\begin{pmatrix} 0 \\ -m_3 \\ m_2 \end{pmatrix} \quad \text{and we note that} \quad \iiint_V \nabla\times\vec{F}\, dV=\begin{pmatrix} 0 \\ -m_3 V \\ m_2 V \end{pmatrix}. \tag{60}$$

On the other hand, since $\vec{m}\times\vec{F}=0$ we have $\iint_{plane} \vec{ds}\times\vec{F}=0$. In view of the equation (59) and using the $y'-$ component of the integral, we obtain

$$Volume=\frac{1}{m_3}[\iint_S \vec{F}\times\vec{ds}]_{y'}=\iint_S x'\,dy'\,dz'-\frac{m_1}{m_3}x'\,dx'\,dy'. \tag{61}$$

We now apply the equation (61) to find the volume bounded by the quadric and the plane as follows:

Example 7 *We consider the scenario of the volume discussed in* ***Example 1****. As a result of the calculations we have the volume of the solid* S_1 *to be about* 20.80907893464 *in both cases.*(*See Maple worksheet in* [9])

Example 8 *We consider the scenario of the volume discussed in* ***Example 2****. With the change of variables using the spherical coordinates we get*

$$\frac{V}{abc}=\frac{1}{3}\int_0^{2\pi}(\cos^2\varphi(2+\cos\theta(\varphi)\sin^2\theta(\varphi)+2\cos\theta(\varphi))+\frac{m_1 a}{m_3 c}\sin^3\theta(\varphi)\cos\varphi)\,d\varphi. \tag{62}$$

Similarly, we may use the $z'-$ *component for computing the integral for equation* (60). *We substitute* $\vec{ds}$ and *get*

$$V=\frac{1}{m_2}[\iint_S \vec{F}\times\vec{ds}]_{z'}=\iint_S \frac{m_1}{m_2}x'\,dz'\,dx'-x'\,dy'\,dz'. \tag{63}$$

Moreover, we have

$$\frac{V}{abc}=-\frac{2}{3}\int_0^{2\pi}(1+\sin^2\theta(\varphi)\cos\theta(\varphi)+\cos\theta(\varphi))\cos^2\varphi\,d\varphi+$$

$$+\frac{m_1 a}{3m_2 b}\int_0^{2\pi}(2+\cos\theta(\varphi)\sin^2\theta(\varphi)+2\cos\theta(\varphi))\sin\varphi\cos\varphi\,d\varphi. \tag{64}$$

With the help of Maple, we perform numerical calculations when both components are used. We obtain the volume of S_2 *to be about* 50.496501830226 *when* $y'-$ *component is used for computation and* S_2 *to be about* 50.496501830226 *when* $z'-$ *component is used for computation.* (*See Maple worksheet in* [9]).

Example 9 *We consider the scenario of the volume bounded by the hyperboloid and the plane discussed in* ***Example 3****. We follow the calculation technique from* ***Example 5****, and with the help of Maple, we perform numerical calculations for both components and obtain the volume of the* solid S_3 *to be about 46.*9249144925 *for the first case and* 46.9249144927 *for the second case.* (*See Maple worksheet in* [9]).

4.5. Calculation of the volume using Stokes' Theorem Part 2

The idea of using Stokes' theorem for calculating the volume of a solid is described as follows. We choose a vector field $\vec{F}$ satisfying $\begin{cases}\nabla\times\vec{F}=(0,0,1),\, x''<0,\\ \nabla\times\vec{F}=\vec{0}, x''>0.\end{cases}$ For example, we set

$$\vec{F}=\begin{pmatrix}P\\Q\\0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}y''-x''\\-y''\\0\end{pmatrix}\arcsin\left(\frac{y''}{\sqrt{(x'')^2+(y'')^2}}\right). \tag{65}$$

We observe that

$$\vec{ds}\times\vec{F}=(F_{z''}ds_{y''}-F_{y''}ds_{z''})\vec{i}+(F_{x''}ds_{z''}-F_{z}ds_{x''})\vec{j}+(F_{y''}ds_{x''}-F_{x''}ds_{y''})\vec{k},$$

$$\vec{ds}\times\vec{F}=-Q\,dx''dy''\vec{i}+P\,dx''dy''\vec{j}+(Q\,dy''dz''-P\,dx''dz'')\vec{k},$$

and for the z'' component we get

$$\iiint_v [\nabla\times\vec{F}]_{z''}dx''dy''dz''=\iint_s [\vec{ds}\times\vec{F}]_{z''}=\oiint_s Q\,dy''dz''-P\,dx''dz''. \quad (66)$$

We see that the left-hand side of the equation (66) is equal to the volume of the solid piece to which $x''<0$. In particular, we have the following:

$$\iiint_v \nabla\times\vec{F}\,dx''dy''dz''=\iiint_{v,x''<0} \nabla\times\vec{F}\,dx''dy''dz''=V_{x''<0},$$

and the right side of (66) contains the surface integral over the entire surface of the quadric surface.

We now apply the equations (65) and (66) to find the volume bounded by the quadric and the plane as follows:

Example 10 *We consider the scenario of the volume discussed in* ***Example 2****. We get the volume of the solid* S_2*, a piece of the ellipsoid with* $x''<0$*, by using the formula of*

$$V_{x''<0}=\oiint_{ellipsoid} P(x'',y'')dx''dz''-Q(x'',y'')dy''dz''.$$

As a result of calculation, we obtain the volume of the solid S_2 *to be approximately* 50.496500. *The same method can be applied for calculating the volume of the other piece of the ellipsoid, which is about* 31.181599 (*See Maple worksheet in* [10])

4.6. Calculation of the volume using the distance to the plane

Here we apply an integration method for calculating the volume of the intersecting solid by using the distance from the quadric to the intersecting plane. However, the approach described here needs to be addressed separately for each solid S_1, S_2, and S_3, which we describe as follows:

4.6.1. For the considered solids S_1, S_2, and S_3, it is convenient to choose a coordinate system X'' so that the equation of the quadric looks like $\vec{X}''^T r A r^T \vec{X}''+2(\vec{A}_0^T A+\vec{B})r\vec{X}''+\vec{A}_0^T A\vec{A}_0+2\vec{B}\vec{A}_0+a_{00}=0$, and the equation of the plane is of the form of $x''=0$. The cross section of the solids is the ellipse with the semi–axes $a_e=\sqrt{\frac{a_0''}{l_{y''}}}$, and $b_e=\sqrt{\frac{a_0''}{l_{z''}}}$, respectively. We perform the integration as follows:

$$\int_{section} x''(y'',z'')d\,y''dz''=a_e b_e\int_0^{2\pi} d\,\psi\int_0^1 x''(a_e\cos\psi, b_e\sin\psi)\rho\,d\,\rho. \quad (67)$$

For the solid S_3 the whole solid is inside an elliptical cylinder, whose generator is perpendicular to the plane, see Figure 4 for demonstration. Therefore the calculated integral (67) coincides with the volume of the solid of intersection. As the result of calculations, we see the volume of the solid S_3 to be approximately 46.9249144958. (*See Maple worksheet in* [11]).

For the solids S_1 and S_2, the result of the calculations by the formula (67) is different from the required volume. The given integral allows us to find only the volume of the piece of the solid inside the elliptical cylinder, whose generator is perpendicular to the plane.

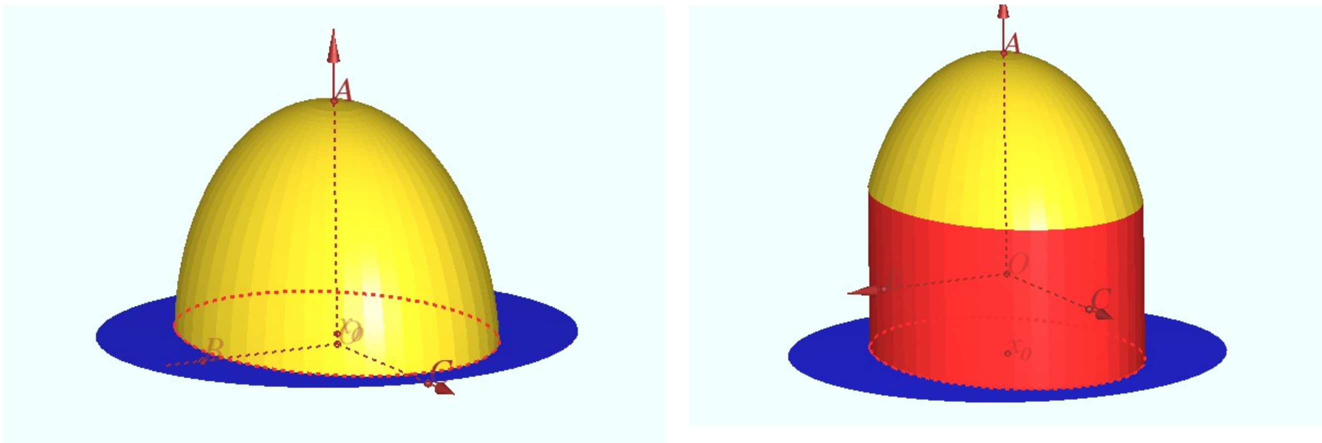

Figure 5. Volume of left solid defined correctly. Volume of right ellipsoid segment defined incorrectly, part of it is outside the elliptical cylinder.

We describe an alternative way of calculating the volume for the bounded solid by using the distance from the solid to the intersecting plane as follows:

4.6.2. Let an arbitrary surface $f(\vec{X}')=0$ be cut by a plane of the form $\vec{n}\cdot\vec{X}'+n_0=0$, where $|\vec{n}|=1$. We would like to find the volume of the solid bounded by the surface and the plane.

We assume the distance from point $\vec{X}'$ to the plane, in other words, the length of the perpendicular dropped from point $\vec{X}'$ onto the plane, is equal to $D=\vec{n}\cdot\vec{X}'+n_0$. The projection of the elementary area $\vec{dS}$ of the surface $f(\vec{X}')=0$ on the plane is equal to $\vec{n}\cdot\vec{dS}$. Therefore, the volume bounded by the surface and the plane can be calculated by using the following formula

$$V=\iint\limits_{surface}(\vec{n}\cdot\vec{X}'+n_0)\vec{n}\cdot\vec{ds}=\iint\limits_{surface}(n_{x'}x'+n_{y'}y'+n_{z'}z'+n_0)(n_{x'}dy'dz'+n_{y'}dx'dz'+n_{z'}dx'dy'). \quad (68)$$

The calculations have been performed in the same way as in Section 4.4. As a result we obtained that the volume of the solid S_1 is approximately 20.809078934644. Similarly, S_2 and S_3 can be calculated and they are approximately 50.496501830224 and 46.9249144928, respectively. (See Maple worksheet in [9]). The resulting formula (67) is similar to a similar one obtained in [2] (see Theorem 12 in [2]).

5 Comparing The Results of Calculations

	Paraboloid S_1	Ellipsoid S_2	Hyperboloid S_3
4.1	20.809078934643	50.496501830226	46.9249144931
4.2	20.8090792	50.4965018300	46.924952
4.3	20.809078934647	50.496501830226	46.9249144927
4.4	20.809078934644	50.496501830226	46.9249144926
	20.809078934645	50.496501830226	46.9249144927
4.5	–	50.496500	–
4.6.1	–	–	46.9249144958
4.6.2	20.809078934644	50.496501830224	46.9249144928

6 Conclusion

In this paper, we summarize several methods of calculating the volume of a solid bounded by a quadric surface and a plane. Through the combination use of the CAS, Maple, (see [4]) and the geometric software, GInMA (see [3]), we are able not only to perform complex computations algebraically but also visualize if our algebraic solutions coincide with our theories analytically. Through advanced and evolving technological tools, students and researchers are able to explore more challenging real-life problems.

References

[1] G.A.Korn, and T.M.Korn. Mathematical Handbook for Scientists and Engineers. Definitions, Theorems, and Formulas for Reference and Review. DOVER PUBLICATIONS, INC. Mineola, New York.

[2] W.–C.Yang, and M.–L.Lo. *Finding Signed Areas and Volumes Inspired by Technology.*, Electronic Journal of Mathematics and Technology (eJMT), ISSN 1933-2823, Issue 2, Vol. 2, 2008.

[3] W.–C.Yang, and V.Shelomovskii. *Mean Value Theorems in Higher Dimensions and Their Applications.*, Electronic Journal of Mathematics and Technology (eJMT), Issue 1, Vol.6, 2012.

[4] X. Shengxiang, W.–C.Yang, and V.Shelomovskii. *Computing Signed Areas and Volumes with Maple.*, Electronic Journal of Mathematics and Technology (eJMT), Issue 2, Vol.6, 2012.

Software Packages

[5] [GInMA] GInMA, 2012, Nosulya, S., Shelomovskii, D. and Shelomovskii, V.
http://deoma–cmd.ru/en/Products/Geometry/GInMA.aspx

[6] [Maple] A product of Maplesoft, http://www.maplesoft.com/.

Supplemental Electronic Materials

All GInMA supplemental materials that accompany this paper can be used from figures.
Install the GInMA software from the website and click on the picture.

[7] V.Shelomovskii. Maple worksheet for Examples 1-3. This worksheet **may take up to** 250 seconds to run.

[8] V.Shelomovskii. Maple worksheet for section 4.2.

[9] V.Shelomovskii. Maple worksheet for sections 4.1,4.3,4.4,4.6.2. This worksheet **may take up to** 250 seconds to run.

[10] V.Shelomovskii. Maple worksheet for section 4.5. This worksheet **may take up to** 520 seconds to run.

[11] V.Shelomovskii. Maple worksheet for section 4.6.1.